4 Line and Surface Integrals

4.1 Line Integrals

In single-variable calculus you learned how to integrate a real-valued function $f(x)$ over an interval $[a,b]$ in $\mathbb{R}^1$. This integral (usually called a *Riemann integral*) can be thought of as an integral over a *path* in $\mathbb{R}^1$, since an interval (or collection of intervals) is really the only kind of "path" in $\mathbb{R}^1$. You may also recall that if $f(x)$ represented the force applied along the x-axis to an object at position x in $[a,b]$, then the *work* W done in moving that object from position $x=a$ to $x=b$ was defined as the integral:

$$W = \int_a^b f(x)\,dx$$

In this section, we will see how to define the integral of a function (either real-valued or vector-valued) of two variables over a general path (i.e. a curve) in $\mathbb{R}^2$. This definition will be motivated by the physical notion of work. We will begin with real-valued functions of two variables.

In physics, the intuitive idea of work is that

$$\text{Work} = \text{Force} \times \text{Distance}\,.$$

Suppose that we want to find the total amount W of work done in moving an object along a curve C in $\mathbb{R}^2$ with a smooth parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, with a force $f(x,y)$ which varies with the position (x,y) of the object and is applied in the direction of motion along C (see Figure 4.1.1 below).

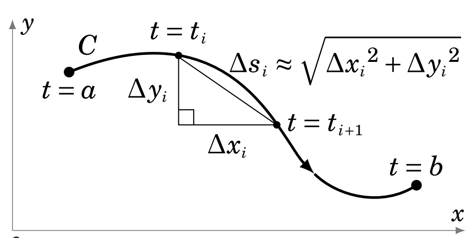


Figure 4.1.1 Curve $C: x = x(t), y = y(t)$ for t in $[a,b]$

We will assume for now that the function $f(x,y)$ is continuous and real-valued, so we only consider the magnitude of the force. Partition the interval $[a,b]$ as follows:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b\ ,\quad \text{for some integer } n \ge 2$$

As we can see from Figure 4.1.1, over a typical subinterval $[t_i, t_{i+1}]$ the distance Δs_i traveled along the curve is approximately $\sqrt{\Delta x_i^2 + \Delta y_i^2}$, by the Pythagorean Theorem. Thus, if the subinterval is small enough then the work done in moving the object along that piece of the curve is approximately

$$\text{Force} \times \text{Distance} \approx f(x_{i*}, y_{i*}) \sqrt{\Delta x_i^2 + \Delta y_i^2}\ , \tag{4.1}$$

where $(x_{i*}, y_{i*}) = (x(t_i*), y(t_i*))$ for some t_i* in $[t_i, t_{i+1}]$, and so

$$W \approx \sum_{i=0}^{n-1} f(x_{i*}, y_{i*}) \sqrt{\Delta x_i^2 + \Delta y_i^2} \tag{4.2}$$

is approximately the total amount of work done over the entire curve. But since

$$\sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i\ ,$$

where $\Delta t_i = t_{i+1} - t_i$, then

$$W \approx \sum_{i=0}^{n-1} f(x_{i*}, y_{i*}) \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i\ . \tag{4.3}$$

Taking the limit of that sum as the length of the largest subinterval goes to 0, the sum over all subintervals becomes the integral from $t = a$ to $t = b$, $\frac{\Delta x_i}{\Delta t_i}$ and $\frac{\Delta y_i}{\Delta t_i}$ become $x'(t)$ and $y'(t)$, respectively, and $f(x_{i*}, y_{i*})$ becomes $f(x(t), y(t))$, so that

$$W = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt\ . \tag{4.4}$$

The integral on the right side of the above equation gives us our idea of how to define, for *any* real-valued function $f(x, y)$, the integral of $f(x, y)$ along the curve C, called a *line integral*:

Definition 4.1. For a real-valued function $f(x, y)$ and a curve C in $\mathbb{R}^2$, parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$, the **line integral of $f(x, y)$ along C with respect to arc length** s is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt\ . \tag{4.5}$$

The symbol ds is the differential of the arc length function

$$s = s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2}\, du\ , \tag{4.6}$$

which you may recognize from Section 1.9 as the length of the curve C over the interval $[a,t]$, for all t in $[a,b]$. That is,

$$ds = s'(t)\,dt = \sqrt{x'(t)^2 + y'(t)^2}\,dt\,, \tag{4.7}$$

by the Fundamental Theorem of Calculus.

For a general real-valued function $f(x,y)$, what does the line integral $\int_C f(x,y)\,ds$ represent? The preceding discussion of ds gives us a clue. You can think of differentials as infinitesimal lengths. So if you think of $f(x,y)$ as the height of a picket fence along C, then $f(x,y)\,ds$ can be thought of as approximately the area of a section of that fence over some infinitesimally small section of the curve, and thus the line integral $\int_C f(x,y)\,ds$ is the total area of that picket fence (see Figure 4.1.2).

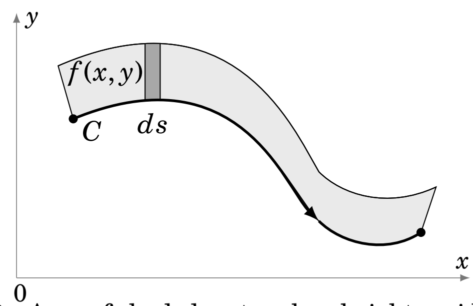


Figure 4.1.2 Area of shaded rectangle = height × width ≈ $f(x,y)\,ds$

Example 4.1. Use a line integral to show that the lateral surface area A of a right circular cylinder of radius r and height h is $2\pi rh$.

Solution: We will use the right circular cylinder with base circle C given by $x^2 + y^2 = r^2$ and with height h in the positive z direction (see Figure 4.1.3). Parametrize C as follows:

$$x = x(t) = r\cos t\,, \quad y = y(t) = r\sin t\,, \quad 0 \le t \le 2\pi$$

Let $f(x,y) = h$ for all (x,y). Then

$$\begin{aligned} A &= \int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt \\ &= \int_0^{2\pi} h\sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt \\ &= h\int_0^{2\pi} r\sqrt{\sin^2 t + \cos^2 t}\,dt \\ &= rh\int_0^{2\pi} 1\,dt = 2\pi rh \end{aligned}$$

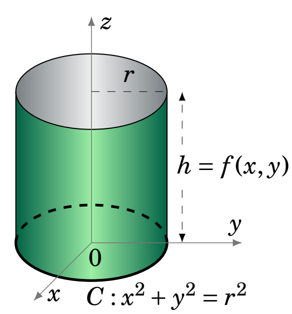


Figure 4.1.3

Note in Example 4.1 that if we had traversed the circle C twice, i.e. let t vary from 0 to 4π, then we would have gotten an area of $4\pi rh$, i.e. twice the desired area, even though the *curve* itself is still the same (namely, a circle of radius r). Also, notice that we traversed the circle in the counter-clockwise direction. If we had gone in the clockwise direction, using the parametrization

$$x = x(t) = r\cos(2\pi - t)\,, \quad y = y(t) = r\sin(2\pi - t)\,, \quad 0 \le t \le 2\pi\,, \tag{4.8}$$

then it is easy to verify (see Exercise 12) that the value of the line integral is unchanged.

In general, it can be shown (see Exercise 15) that reversing the direction in which a curve C is traversed leaves $\int_C f(x,y)\,ds$ unchanged, for any $f(x,y)$. If a curve C has a parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, then denote by $-C$ the same curve as C but traversed in the opposite direction. Then $-C$ is parametrized by

$$x = x(a+b-t)\,, \quad y = y(a+b-t)\,, \quad a \le t \le b\,, \tag{4.9}$$

and we have

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds\,. \tag{4.10}$$

Notice that our definition of the line integral was with respect to the arc length parameter s. We can also define

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt \tag{4.11}$$

as the *line integral of* $f(x,y)$ *along* C *with respect to* x, and

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt \tag{4.12}$$

as the *line integral of* $f(x,y)$ *along* C *with respect to* y.

In the derivation of the formula for a line integral, we used the idea of work as force multiplied by distance. However, we know that force is actually a *vector*. So it would be helpful to develop a vector form for a line integral. For this, suppose that we have a function $\mathbf{f}(x,y)$ defined on $\mathbb{R}^2$ by

$$\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

for some continuous real-valued functions $P(x,y)$ and $Q(x,y)$ on $\mathbb{R}^2$. Such a function $\mathbf{f}$ is called a **vector field** on $\mathbb{R}^2$. It is defined at *points* in $\mathbb{R}^2$, and its values are *vectors* in $\mathbb{R}^2$. For a curve C with a smooth parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

be the position vector for a point $(x(t), y(t))$ on C. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$ and so

$$\begin{aligned}\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy &= \int_a^b P(x(t),y(t))\,x'(t)\,dt + \int_a^b Q(x(t),y(t))\,y'(t)\,dt \\ &= \int_a^b \big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\big)\,dt \\ &= \int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)\,dt\end{aligned}$$

by definition of $\mathbf{f}(x, y)$. Notice that the function $\mathbf{f}(x(t), y(t)) \cdot \mathbf{r}'(t)$ is a *real-valued* function on $[a, b]$, so the last integral on the right looks somewhat similar to our earlier definition of a line integral. This leads us to the following definition:

Definition 4.2. For a vector field $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and a curve C with a smooth parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, the **line integral of f along** C is

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy \tag{4.13}$$

$$= \int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)\,dt\,, \tag{4.14}$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is the position vector for points on C.

We use the notation $d\mathbf{r} = \mathbf{r}'(t)\,dt = dx\,\mathbf{i} + dy\,\mathbf{j}$ to denote the **differential** of the vector-valued function $\mathbf{r}$. The line integral in Definition 4.2 is often called a *line integral of a vector field* to distinguish it from the line integral in Definition 4.1 which is called a *line integral of a scalar field*. For convenience we will often write

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy\,,$$

where it is understood that the line integral along C is being applied to both P and Q. The quantity $P(x, y)\,dx + Q(x, y)\,dy$ is known as a **differential form**. For a real-valued function $F(x, y)$, the **differential** of F is $dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$. A differential form $P(x, y)\,dx + Q(x, y)\,dy$ is called **exact** if it equals dF for some function $F(x, y)$.

Recall that if the points on a curve C have position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{r}'(t)$ is a tangent vector to C at the point $(x(t), y(t))$ in the direction of increasing t (which we call the *direction of* C). Since C is a smooth curve, then $\mathbf{r}'(t) \neq \mathbf{0}$ on $[a, b]$ and hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

is the unit tangent vector to C at $(x(t), y(t))$. Putting Definitions 4.1 and 4.2 together we get the following theorem:

Theorem 4.1. For a vector field $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ and a curve C with a smooth parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$ and position vector $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$,

$$\int_C \mathbf{f} \cdot d\mathbf{r} \;=\; \int_C \mathbf{f} \cdot \mathbf{T}\, ds\,, \tag{4.15}$$

where $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ is the unit tangent vector to C at $(x(t), y(t))$.

If the vector field $\mathbf{f}(x,y)$ represents the force moving an object along a curve C, then the work W done by this force is

$$W \;=\; \int_C \mathbf{f} \cdot \mathbf{T}\, ds \;=\; \int_C \mathbf{f} \cdot d\mathbf{r}\,. \tag{4.16}$$

Example 4.2. Evaluate $\int_C (x^2 + y^2)\,dx + 2xy\,dy$, where:

(a) $C : x = t\,, \quad y = 2t\,, \quad 0 \le t \le 1$

(b) $C : x = t\,, \quad y = 2t^2\,, \quad 0 \le t \le 1$

Solution: Figure 4.1.4 shows both curves.

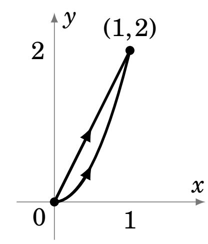


Figure 4.1.4

(a) Since $x'(t) = 1$ and $y'(t) = 2$, then

$$\begin{aligned}
\int_C (x^2 + y^2)\,dx + 2xy\,dy &= \int_0^1 \big((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)\,y'(t)\big)\,dt \\
&= \int_0^1 \big((t^2 + 4t^2)(1) + 2t(2t)(2)\big)\,dt \\
&= \int_0^1 13t^2\,dt \\
&= \left.\frac{13t^3}{3}\right|_0^1 \;=\; \frac{13}{3}
\end{aligned}$$

(b) Since $x'(t) = 1$ and $y'(t) = 4t$, then

$$\begin{aligned}
\int_C (x^2 + y^2)\,dx + 2xy\,dy &= \int_0^1 \big((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)\,y'(t)\big)\,dt \\
&= \int_0^1 \big((t^2 + 4t^4)(1) + 2t(2t^2)(4t)\big)\,dt \\
&= \int_0^1 (t^2 + 20t^4)\,dt \\
&= \left.\frac{t^3}{3} + 4t^5\right|_0^1 \;=\; \frac{1}{3} + 4 \;=\; \frac{13}{3}
\end{aligned}$$

So in both cases, if the vector field $\mathbf{f}(x, y) = (x^2 + y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ represents the force moving an object from $(0,0)$ to $(1,2)$ along the given curve C, then the work done is $\frac{13}{3}$. This may lead you to think that work (and more generally, the line integral of a vector field) is independent of the path taken. However, as we will see in the next section, this is not always the case.

Although we defined line integrals over a single smooth curve, if C is a *piecewise smooth curve*, that is

$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$

is the union of smooth curves $C_1, \ldots, C_n$, then we can define

$$\int_C \mathbf{f} \cdot d\mathbf{r} \;=\; \int_{C_1} \mathbf{f} \cdot d\mathbf{r}_1 \;+\; \int_{C_2} \mathbf{f} \cdot d\mathbf{r}_2 \;+ \ldots +\; \int_{C_n} \mathbf{f} \cdot d\mathbf{r}_n$$

where each $\mathbf{r}_i$ is the position vector of the curve C_i.

Example 4.3. Evaluate $\int_C (x^2 + y^2)\,dx + 2xy\,dy$, where C is the polygonal path from $(0,0)$ to $(0,2)$ to $(1,2)$.

Solution: Write $C = C_1 \cup C_2$, where C_1 is the curve given by $x = 0$, $y = t$, $0 \le t \le 2$ and C_2 is the curve given by $x = t$, $y = 2$, $0 \le t \le 1$ (see Figure 4.1.5). Then

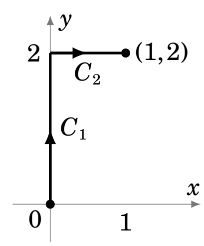


Figure 4.1.5

$$\begin{aligned}
\int_C (x^2 + y^2)\,dx + 2xy\,dy &= \int_{C_1} (x^2 + y^2)\,dx + 2xy\,dy \\
&\quad + \int_{C_2} (x^2 + y^2)\,dx + 2xy\,dy \\
&= \int_0^2 \big((0^2 + t^2)(0) + 2(0)t(1)\big)\,dt \;+\; \int_0^1 \big((t^2 + 4)(1) + 2t(2)(0)\big)\,dt \\
&= \int_0^2 0\,dt + \int_0^1 (t^2 + 4)\,dt \\
&= \frac{t^3}{3} + 4t \,\Bigg|_0^1 \;=\; \frac{1}{3} + 4 \;=\; \frac{13}{3}
\end{aligned}$$

Line integral notation varies quite a bit. For example, in physics it is common to see the notation $\int_a^b \mathbf{f} \cdot d\mathbf{l}$, where it is understood that the limits of integration a and b are for the underlying parameter t of the curve, and the letter $\mathbf{l}$ signifies length. Also, the formulation $\int_C \mathbf{f} \cdot \mathbf{T}\,ds$ from Theorem 4.1 is often preferred in physics since it emphasizes the idea of integrating the tangential component $\mathbf{f} \cdot \mathbf{T}$ of $\mathbf{f}$ in the direction of $\mathbf{T}$ (i.e. in the direction of C), which is a useful physical interpretation of line integrals.

Exercises

A

For Exercises 1-4, calculate $\int_C f(x,y)\,ds$ for the given function $f(x,y)$ and curve C.

1. $f(x,y) = xy$; $\quad C : x = \cos t,\ y = \sin t,\ 0 \le t \le \pi/2$

2. $f(x,y) = \dfrac{x}{x^2+1}$; $\quad C : x = t,\ y = 0,\ 0 \le t \le 1$

3. $f(x,y) = 2x + y$; $\quad C$: polygonal path from $(0,0)$ to $(3,0)$ to $(3,2)$

4. $f(x,y) = x + y^2$; $\quad C$: path from $(2,0)$ counterclockwise along the circle $x^2 + y^2 = 4$ to the point $(-2,0)$ and then back to $(2,0)$ along the x-axis

5. Use a line integral to find the lateral surface area of the part of the cylinder $x^2 + y^2 = 4$ below the plane $x + 2y + z = 6$ and above the xy-plane.

For Exercises 6-11, calculate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for the given vector field $\mathbf{f}(x,y)$ and curve C.

6. $\mathbf{f}(x,y) = \mathbf{i} - \mathbf{j}$; $\quad C : x = 3t,\ y = 2t,\ 0 \le t \le 1$

7. $\mathbf{f}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$; $\quad C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

8. $\mathbf{f}(x,y) = x\,\mathbf{i} + y\,\mathbf{j}$; $\quad C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

9. $\mathbf{f}(x,y) = (x^2 - y)\,\mathbf{i} + (x - y^2)\,\mathbf{j}$; $\quad C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

10. $\mathbf{f}(x,y) = xy^2\,\mathbf{i} + xy^3\,\mathbf{j}$; $\quad C$: the polygonal path from $(0,0)$ to $(1,0)$ to $(0,1)$ to $(0,0)$

11. $\mathbf{f}(x,y) = (x^2 + y^2)\,\mathbf{i}$; $\quad C : x = 2 + \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

B

12. Verify that the value of the line integral in Example 4.1 is unchanged when using the parametrization of the circle C given in formulas (4.8).

13. Show that if $\mathbf{f} \perp \mathbf{r}'(t)$ at each point $\mathbf{r}(t)$ along a smooth curve C, then $\int_C \mathbf{f} \cdot d\mathbf{r} = 0$.

14. Show that if $\mathbf{f}$ points in the same direction as $\mathbf{r}'(t)$ at each point $\mathbf{r}(t)$ along a smooth curve C, then $\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \|\mathbf{f}\|\,ds$.

C

15. Prove that $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$. (*Hint: Use formulas (4.9).*)

16. Let C be a smooth curve with arc length L, and suppose that $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is a vector field such that $\|\mathbf{f}(x,y)\| \le M$ for all (x,y) on C. Show that $\left|\int_C \mathbf{f} \cdot d\mathbf{r}\right| \le ML$. (*Hint: Recall that* $\left|\int_a^b g(x)\,dx\right| \le \int_a^b |g(x)|\,dx$ *for Riemann integrals.*)

17. Prove that the Riemann integral $\int_a^b f(x)\,dx$ is a special case of a line integral.

4.2 Properties of Line Integrals

We know from the previous section that for line integrals of real-valued functions (scalar fields), reversing the direction in which the integral is taken along a curve does not change the value of the line integral:

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds \tag{4.17}$$

For line integrals of vector fields, however, the value does change. To see this, let $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ be a vector field, with P and Q continuously differentiable functions. Let C be a smooth curve parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$, with position vector $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ (we will usually abbreviate this by saying that $C : \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a smooth curve). We know that the curve $-C$ traversed in the opposite direction is parametrized by $x = x(a+b-t)$, $y = y(a+b-t)$, $a \le t \le b$. Then

$$\begin{aligned}
\int_{-C} P(x,y)\,dx &= \int_a^b P(x(a+b-t), y(a+b-t))\,\frac{d}{dt}(x(a+b-t))\,dt \\
&= \int_a^b P(x(a+b-t), y(a+b-t))(-x'(a+b-t))\,dt \quad \text{(by the Chain Rule)} \\
&= \int_b^a P(x(u), y(u))(-x'(u))(-du) \quad \text{(by letting } u = a+b-t\text{)} \\
&= \int_b^a P(x(u), y(u))\,x'(u)\,du \\
&= -\int_a^b P(x(u), y(u))\,x'(u)\,du\ , \quad \text{since } \int_b^a = -\int_a^b\ , \text{ so} \\
\int_{-C} P(x,y)\,dx &= -\int_C P(x,y)\,dx
\end{aligned}$$

since we are just using a different letter (u) for the line integral along C. A similar argument shows that

$$\int_{-C} Q(x,y)\,dy = -\int_C Q(x,y)\,dy\ ,$$

and hence

$$\begin{aligned}
\int_{-C} \mathbf{f}\cdot d\mathbf{r} &= \int_{-C} P(x,y)\,dx + \int_{-C} Q(x,y)\,dy \\
&= -\int_C P(x,y)\,dx + -\int_C Q(x,y)\,dy \\
&= -\left(\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy\right) \\
\int_{-C} \mathbf{f}\cdot d\mathbf{r} &= -\int_C \mathbf{f}\cdot d\mathbf{r}\ .
\end{aligned} \tag{4.18}$$

The above formula can be interpreted in terms of the work done by a force $\mathbf{f}(x,y)$ (treated as a vector) moving an object along a curve C: the total work performed moving the object along C from its initial point to its terminal point, and then back to the initial point moving backwards along the same path, is zero. This is because when force is considered as a vector, direction is accounted for.

The preceding discussion shows the importance of always taking the *direction* of the curve into account when using line integrals of vector fields. For this reason, the curves in line integrals are sometimes referred to as *directed curves* or *oriented curves*.

Recall that our definition of a line integral required that we have *a* parametrization $x = x(t),\ y = y(t),\ a \le t \le b$ for the curve C. But as we know, any curve has infinitely many parametrizations. So could we get a different value for a line integral using some other parametrization of C, say, $x = \tilde{x}(u),\ y = \tilde{y}(u),\ c \le u \le d$? If so, this would mean that our definition is not well-defined. Luckily, it turns out that the value of a line integral of a vector field is unchanged as long as the direction of the curve C is preserved by whatever parametrization is chosen:

Theorem 4.2. Let $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field, and let C be a smooth curve parametrized by $x = x(t),\ y = y(t),\ a \le t \le b$. Suppose that $t = \alpha(u)$ for $c \le u \le d$, such that $a = \alpha(c)$, $b = \alpha(d)$, and $\alpha'(u) > 0$ on the open interval (c,d) (i.e. $\alpha(u)$ is strictly increasing on $[c,d]$). Then $\int_C \mathbf{f}\cdot d\mathbf{r}$ has the same value for the parametrizations $x = x(t),\ y = y(t),\ a \le t \le b$ and $x = \tilde{x}(u) = x(\alpha(u)),\ y = \tilde{y}(u) = y(\alpha(u)),\ c \le u \le d$.

Proof: Since $\alpha(u)$ is strictly increasing and maps $[c,d]$ onto $[a,b]$, then we know that $t = \alpha(u)$ has an inverse function $u = \alpha^{-1}(t)$ defined on $[a,b]$ such that $c = \alpha^{-1}(a)$, $d = \alpha^{-1}(b)$, and $\frac{du}{dt} = \frac{1}{\alpha'(u)}$. Also, $dt = \alpha'(u)\,du$, and by the Chain Rule

$$\tilde{x}'(u) = \frac{d\tilde{x}}{du} = \frac{d}{du}(x(\alpha(u))) = \frac{dx}{dt}\frac{dt}{du} = x'(t)\,\alpha'(u) \quad\Rightarrow\quad x'(t) = \frac{\tilde{x}'(u)}{\alpha'(u)}$$

so making the susbstitution $t = \alpha(u)$ gives

$$\begin{aligned}\int_a^b P(x(t),y(t))\,x'(t)\,dt &= \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} P(x(\alpha(u)),y(\alpha(u)))\,\frac{\tilde{x}'(u)}{\alpha'(u)}\,(\alpha'(u)\,du)\\ &= \int_c^d P(\tilde{x}(u),\tilde{y}(u))\,\tilde{x}'(u)\,du\ ,\end{aligned}$$

which shows that $\int_C P(x,y)\,dx$ has the same value for both parametrizations. A similar argument shows that $\int_C Q(x,y)\,dy$ has the same value for both parametrizations, and hence $\int_C \mathbf{f}\cdot d\mathbf{r}$ has the same value. **QED**

Notice that the condition $\alpha'(u) > 0$ in Theorem 4.2 means that the two parametrizations move along C in the same direction. That was *not* the case with the "reverse" parametrization for $-C$: for $u = a + b - t$ we have $t = \alpha(u) = a + b - u \Rightarrow \alpha'(u) = -1 < 0$.

Example 4.4. Evaluate the line integral $\int_C (x^2+y^2)\,dx + 2xy\,dy$ from Example 4.2, Section 4.1, along the curve $C : x = t,\ y = 2t^2,\ 0 \le t \le 1$, where $t = \sin u$ for $0 \le u \le \pi/2$.

Solution: First, we notice that $0 = \sin 0$, $1 = \sin(\pi/2)$, and $\frac{dt}{du} = \cos u > 0$ on $(0, \pi/2)$. So by Theorem 4.2 we know that if C is parametrized by

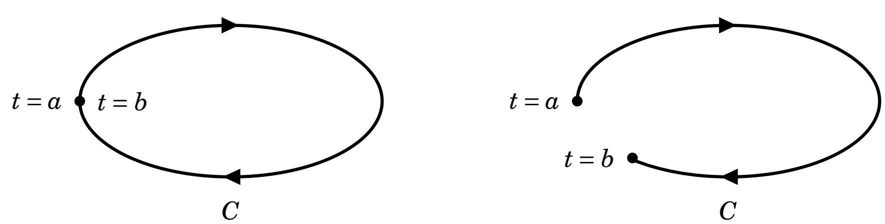
$$x = \sin u\ , \quad y = 2\sin^2 u\ , \quad 0 \le u \le \pi/2$$

then $\int_C (x^2+y^2)\,dx + 2xy\,dy$ should have the same value as we found in Example 4.2, namely $\frac{13}{3}$. And we can indeed verify this:

$$\begin{aligned}\int_C (x^2+y^2)\,dx + 2xy\,dy &= \int_0^{\pi/2} \left((\sin^2 u + (2\sin^2 u)^2)\cos u + 2(\sin u)(2\sin^2 u)4\sin u\,\cos u\right) du \\ &= \int_0^{\pi/2} \left(\sin^2 u + 20\sin^4 u\right)\cos u\,du \\ &= \frac{\sin^3 u}{3} + 4\sin^5 u\,\Bigg|_0^{\pi/2} \\ &= \frac{1}{3} + 4 \;=\; \frac{13}{3}\end{aligned}$$

In other words, the line integral is unchanged whether t or u is the parameter for C.

By a **closed curve**, we mean a curve C whose initial point and terminal point are the same, i.e. for $C : x = x(t),\ y = y(t),\ a \le t \le b$, we have $(x(a), y(a)) = (x(b), y(b))$.



(a) Closed **(b)** Not closed

Figure 4.2.1 Closed vs nonclosed curves

A **simple closed curve** is a closed curve which does not intersect itself. Note that any closed curve can be regarded as a union of simple closed curves (think of the loops in a figure eight). We use the special notation

$$\oint_C f(x,y)\,ds \quad \text{and} \quad \oint_C \mathbf{f}\cdot d\mathbf{r}$$

to denote line integrals of scalar and vector fields, respectively, along closed curves. In some older texts you may see the notation $\displaystyle\ointctrclockwise$ or $\displaystyle\ointclockwise$ to indicate a line integral traversing a closed curve in a counterclockwise or clockwise direction, respectively.

So far, the examples we have seen of line integrals (e.g. Example 4.2) have had the same value for different curves joining the initial point to the terminal point. That is, the line integral has been independent of the path joining the two points. As we mentioned before, this is not always the case. The following theorem gives a necessary and sufficient condition for this *path independence*:

Theorem 4.3. In a region R, the line integral $\int_C \mathbf{f}\cdot d\mathbf{r}$ is independent of the path between any two points in R if and only if $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for every closed curve C which is contained in R.

Proof: Suppose that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for every closed curve C which is contained in R. Let P_1 and P_2 be two distinct points in R. Let C_1 be a curve in R going from P_1 to P_2, and let C_2 be another curve in R going from P_1 to P_2, as in Figure 4.2.2.

Then $C = C_1 \cup -C_2$ is a closed curve in R (from P_1 to P_1), and so $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$. Thus,

$$\begin{aligned} 0 &= \oint_C \mathbf{f}\cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{f}\cdot d\mathbf{r} + \int_{-C_2} \mathbf{f}\cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{f}\cdot d\mathbf{r} - \int_{C_2} \mathbf{f}\cdot d\mathbf{r}\,,\ \text{and so} \end{aligned}$$

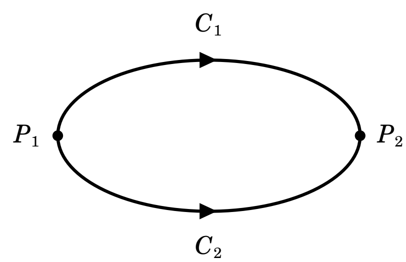


Figure 4.2.2

$\int_{C_1} \mathbf{f}\cdot d\mathbf{r} = \int_{C_2} \mathbf{f}\cdot d\mathbf{r}$. This proves path independence.

Conversely, suppose that the line integral $\int_C \mathbf{f}\cdot d\mathbf{r}$ is independent of the path between any two points in R. Let C be a closed curve contained in R. Let P_1 and P_2 be two distinct points on C. Let C_1 be a part of the curve C that goes from P_1 to P_2, and let C_2 be the remaining part of C that goes from P_1 to P_2, again as in Figure 4.2.2. Then by path independence we have

$$\begin{aligned} \int_{C_1} \mathbf{f}\cdot d\mathbf{r} &= \int_{C_2} \mathbf{f}\cdot d\mathbf{r} \\ \int_{C_1} \mathbf{f}\cdot d\mathbf{r} - \int_{C_2} \mathbf{f}\cdot d\mathbf{r} &= 0 \\ \int_{C_1} \mathbf{f}\cdot d\mathbf{r} + \int_{-C_2} \mathbf{f}\cdot d\mathbf{r} &= 0\,,\ \text{so} \\ \oint_C \mathbf{f}\cdot d\mathbf{r} &= 0 \end{aligned}$$

since $C = C_1 \cup -C_2$. **QED**

Clearly, the above theorem does not give a practical way to determine path independence, since it is impossible to check the line integrals around all possible closed curves in a region. What it mostly does is give an idea of the way in which line integrals behave, and how seemingly unrelated line integrals can be related (in this case, a specific line integral between two points and *all* line integrals around closed curves).

For a more practical method for determining path independence, we first need a version of the Chain Rule for multivariable functions:

Theorem 4.4. (**Chain Rule**) If $z = f(x,y)$ is a continuously differentiable function of x and y, and both $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then z is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{4.19}$$

at all points where the derivatives on the right are defined.

The proof is virtually identical to the proof of Theorem 2.2 from Section 2.4 (which uses the Mean Value Theorem), so we omit it.[1] We will now use this Chain Rule to prove the following *sufficient* condition for path independence of line integrals:

Theorem 4.5. Let $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field in some region R, with P and Q continuously differentiable functions on R. Let C be a smooth curve in R parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$. Suppose that there is a real-valued function $F(x,y)$ such that $\nabla F = \mathbf{f}$ on R. Then

$$\int_C \mathbf{f}\cdot d\mathbf{r} = F(B) - F(A)\,, \tag{4.20}$$

where $A = (x(a), y(a))$ and $B = (x(b), y(b))$ are the endpoints of C. Thus, the line integral is independent of the path between its endpoints, since it depends only on the values of F at those endpoints.

Proof: By definition of $\int_C \mathbf{f}\cdot d\mathbf{r}$, we have

$$\begin{aligned}
\int_C \mathbf{f}\cdot d\mathbf{r} &= \int_a^b \big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\big)\,dt \\
&= \int_a^b \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}\right)dt \quad \left(\text{since } \nabla F = \mathbf{f} \Rightarrow \frac{\partial F}{\partial x} = P \text{ and } \frac{\partial F}{\partial y} = Q\right) \\
&= \int_a^b F'(x(t),y(t))\,dt \quad (\text{by the Chain Rule in Theorem 4.4}) \\
&= F(x(t),y(t))\Big|_a^b = F(B) - F(A)
\end{aligned}$$

by the Fundamental Theorem of Calculus. **QED**

[1] See TAYLOR and MANN, § 6.5.

Theorem 4.5 can be thought of as the line integral version of the Fundamental Theorem of Calculus. A real-valued function $F(x,y)$ such that $\nabla F(x,y) = \mathbf{f}(x,y)$ is called a **potential** for $\mathbf{f}$. A **conservative** vector field is one which has a potential.

Example 4.5. Recall from Examples 4.2 and 4.3 in Section 4.1 that the line integral $\int_C (x^2+y^2)\,dx + 2xy\,dy$ was found to have the value $\frac{13}{3}$ for three different curves C going from the point $(0,0)$ to the point $(1,2)$. Use Theorem 4.5 to show that this line integral is indeed path independent.

Solution: We need to find a real-valued function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = x^2 + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = 2xy\ .$$

Suppose that $\frac{\partial F}{\partial x} = x^2 + y^2$, Then we must have $F(x,y) = \frac{1}{3}x^3 + xy^2 + g(y)$ for some function $g(y)$. So $\frac{\partial F}{\partial y} = 2xy + g'(y)$ satisfies the condition $\frac{\partial F}{\partial y} = 2xy$ if $g'(y) = 0$, i.e. $g(y) = K$, where K is a constant. Since any choice for K will do (why?), we pick $K = 0$. Thus, a potential $F(x,y)$ for $\mathbf{f}(x,y) = (x^2+y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ exists, namely

$$F(x,y) \;=\; \frac{1}{3}x^3 + xy^2\ .$$

Hence the line integral $\int_C (x^2+y^2)\,dx + 2xy\,dy$ is path independent.
Note that we can also verify that the value of the line integral of $\mathbf{f}$ along any curve C going from $(0,0)$ to $(1,2)$ will always be $\frac{13}{3}$, since by Theorem 4.5

$$\int_C \mathbf{f}\cdot d\mathbf{r} \;=\; F(1,2) \;-\; F(0,0) \;=\; \frac{1}{3}(1)^3 + (1)(2)^2 - (0+0) \;=\; \frac{1}{3} + 4 \;=\; \frac{13}{3}\ .$$

A consequence of Theorem 4.5 in the special case where C is a closed curve, so that the endpoints A and B are the same point, is the following important corollary:

Corollary 4.6. If a vector field $\mathbf{f}$ has a potential in a region R, then $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for any closed curve C in R (i.e. $\oint_C \nabla F \cdot d\mathbf{r} = 0$ for any real-valued function $F(x,y)$).

Example 4.6. Evaluate $\oint_C x\,dx + y\,dy$ for $C : x = 2\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$.

Solution: The vector field $\mathbf{f}(x,y) = x\,\mathbf{i} + y\,\mathbf{j}$ has a potential $F(x,y)$:

$$\begin{aligned}
\frac{\partial F}{\partial x} = x \;\Rightarrow\; F(x,y) \;&=\; \frac{1}{2}x^2 + g(y)\ \text{, so}\\
\frac{\partial F}{\partial y} = y \;\Rightarrow\; g'(y) = y \;\Rightarrow\; g(y) \;&=\; \frac{1}{2}y^2 + K
\end{aligned}$$

for any constant K, so $F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a potential for $\mathbf{f}(x,y)$. Thus,

$$\oint_C x\,dx + y\,dy \;=\; \oint_C \mathbf{f}\cdot d\mathbf{r} \;=\; 0$$

by Corollary 4.6, since the curve C is closed (it is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$).

Exercises

A

1. Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$ for $C: x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

2. Evaluate $\int_C (x^2+y^2)\,dx + 2xy\,dy$ for $C: x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$.

3. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$? If so, find one.

4. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$? If so, find one.

5. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = xy^2\,\mathbf{i} + x^3y\,\mathbf{j}$? If so, find one.

B

6. Let $\mathbf{f}(x,y)$ and $\mathbf{g}(x,y)$ be vector fields, let a and b be constants, and let C be a curve in $\mathbb{R}^2$. Show that

$$\int_C (a\,\mathbf{f} \pm b\,\mathbf{g})\cdot d\mathbf{r} \;=\; a\int_C \mathbf{f}\cdot d\mathbf{r} \;\pm\; b\int_C \mathbf{g}\cdot d\mathbf{r}\,.$$

7. Let C be a curve whose arc length is L. Show that $\int_C 1\,ds = L$.

8. Let $f(x,y)$ and $g(x,y)$ be continuously differentiable real-valued functions in a region R. Show that

$$\oint_C f\,\nabla g\cdot d\mathbf{r} \;=\; -\oint_C g\,\nabla f\cdot d\mathbf{r}$$

for any closed curve C in R. (*Hint: Use Exercise 21 in Section 2.4.*)

9. Let $\mathbf{f}(x,y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ for all $(x,y) \ne (0,0)$, and $C: x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

(a) Show that $\mathbf{f} = \nabla F$, for $F(x,y) = \tan^{-1}(y/x)$.

(b) Show that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 2\pi$. Does this contradict Corollary 4.6? Explain.

C

10. Let $g(x)$ and $h(y)$ be differentiable functions, and let $\mathbf{f}(x,y) = h(y)\,\mathbf{i} + g(x)\,\mathbf{j}$. Can $\mathbf{f}$ have a potential $F(x,y)$? If so, find it. You may assume that F would be smooth. (*Hint: Consider the mixed partial derivatives of F.*)